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Spectral gaps of the periodic Schrödinger operator when its potential is an entire function

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Abstract

Consider the Schrödinger equation $-y'' + v(x)y = \lambda y$ with periodic complex-valued potential, of period 1, $v(x) = \sum_{m=-\infty}^{\infty} V(2m)\exp(2\pi \mathrm{i} mx)$. Let λ_n^+, λ_n^-, and μ_n be the eigenvalues of L that are close to $\pi^2 n^2$, respectively with periodic (for n even), antiperiodic (for n odd), and Dirichlet boundary conditions on $[0, 1]$, and let d_n be the diameter of *the spectral triangle* with vertices $\lambda_n^+, \lambda_n^-, \mu_n$. We study the relationship between the rate of decay of "gap sequence" (d_n) and the rate of decay of the sequence of Fourier coefficients $(V(m))$ in the case $v(x)$ is an entire function (then $(V(m))$ decays superexponentially). It is proven that if $|V(n)|\exp(a|n|^b) \in \ell^\infty$ with $a > 0$ and $b > 1$ then $(d_n \exp(c\,n(\log n)^{1-1/b})) \in \ell^\infty$ for some $c > 0$. A special example shows that up to a change of types a, c, this statement is sharp.

1. Introduction

We consider the Schrödinger equation

$$-y'' + vy = \lambda y \tag{1}$$

with a complex-valued periodic L^2-potential $v(x)$ of period 1,

$$v(x) = \sum_{m=-\infty}^{\infty} V(2m)\exp(2\pi \mathrm{i} mx). \tag{2}$$

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If the potential $v(x)$ is real-valued, then the Schrödinger operator $Ly = -y'' + vy$, considered in $L^2(\mathbb{R})$, is an unbounded selfadjoint operator. Its spectrum is a subset of the real line $(-\infty, \infty)$ with a special structure: near each of the points $\pi^2 n^2$ there are spectral gaps, maybe empty $(\lambda_n^-, \lambda_n^+)$, or zones of instability, with $\{\lambda_n^-, \lambda_n^+\}$ being pairs of eigenvalues of (1) on $[0, 1]$ with periodic if n is even and antiperiodic if n is odd boundary conditions. It turns out that the rate of decay of the sequence of spectral gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$ is closely related to the smoothness of v.

If v is complex valued then the Schrödinger operator L is not selfadjoint, so its spectrum is not real anymore. Nevertheless, if n is large enough, then near each of the points $\pi^2 n^2$ there is a pair of complex eigenvalues $\{\lambda_n^-, \lambda_n^+\}$ of (1) on $[0, 1]$ with periodic (if n is even) and antiperiodic (if n is odd) boundary conditions. And, again there is a relationship between the rate of decay of the sequence $\gamma_n = \lambda_n^+ - \lambda_n^-$ and the smoothness of the potential v.

H. Hochstadt [7] observed that a real-valued L^2-potential v is a C^∞-function if and only if the gap sequence

$$\gamma_n = \lambda_n^+ - \lambda_n^-$$

decays faster than any power of $1/n$, that is

$$(\gamma_n) \in \ell_2^N = \left\{ (x_n) \colon \sum |x_n|^2 \left(1 + n^2\right)^N < \infty \right\}$$

for every $N > 0$. Since then the question on relationship between smoothness of v and the decay rate of (γ_n) was discussed in many articles and books. We refer for more information to [10,11,13].

An effective approach (based on Fourier analysis) to this general question was developed recently in [8,9]. In [8] it is proven that if

$$\sum |V(2n)|^2 \omega^2(n) < \infty, \tag{3}$$

where

$$\omega(n) = (1 + |n|)^N \exp(a|n|), \quad N \geqslant 0,\ a > 0, \tag{4}$$

then

$$\sum |\gamma_n|^2 \omega^2(n) < \infty. \tag{5}$$

This result is extended in [9] to all weights ω such that

$$\omega(0) = 1, \qquad \omega(-k) = \omega(k), \quad k \in \mathbb{Z}, \qquad \omega(k) \leqslant \omega(k+1), \quad k \geqslant 0, \tag{6}$$

and ω being *submultiplicative*

$$\omega(k + j) \leqslant \omega(k)\omega(j), \quad \forall k, j \in \mathbb{Z}. \tag{7}$$

Moreover, eigenvalues μ_n of (1) with Dirichlet boundary conditions $y(0) = 0,\ y(1) = 0$ are also studied in [9], and it is proven that (3) for weights ω with (6) and (7) implies

$$\sum \bigl|\mu_n - \bigl(\lambda_n^+ + \lambda_n^-\bigr)/2\bigr|^2 \omega^2(n) < \infty. \tag{8}$$

If v is a real valued potential then (5) implies (3) in the case of Gevrey type weights

$$\omega(n) = (1 + |n|)^s \exp\bigl(an^b\bigr), \quad s, a \geqslant 0,\ b \in (0, 1) \tag{9}$$

—see Theorem 10 in [2].

For complex valued potentials (5) does not imply (3), but in [3] it is proven that (5) and (8) together imply (3).

This paper addresses the same questions, but for weights

$$\Omega(n) = \exp\bigl(a|n|^b\bigr), \quad a > 0,\ b > 1, \tag{10}$$

or, more generally, for weights of the form

$$\Omega(n) = \exp(\varphi(n)), \tag{11}$$

where φ is a convex even function with $\varphi(n)/n \to \infty$. Observe that the main results in [2,8,9] are formulated and proven for subexponential weights, while the weights (10) and (11) are superexponential, i.e.,

$$\limsup \frac{\log \Omega(n)}{n} = \infty.$$

So, roughly speaking, here we study the case where $(V(n))$ decays more rapidly than $\exp(-C|n|)$, $\forall C > 0$. Of course, it is the same to say that we study the case where the potential v is an entire function, since the series (2) converges for any complex x if and only if Fourier coefficients of v decay superexponentially.

2. Preliminaries

Our approach to analysis of the relationship between the decay rates of Fourier coefficients $(V(n))$ of a 1-periodic potential $v(x)$ with $V(0) = \int_0^1 v(x)\,dx = 0$ and the corresponding "gap sequences" $\gamma_n = \lambda_n^+ - \lambda_n^-$ and $\delta_n = \mu_n - (\lambda_n^+ - \lambda_n^-)/2$ follows the Kappeler–Mityagin approach [8,9]. In particular, the fundamental equations (2.5)–(2.7), p. 623, in [8] are very important for us as well; we remind now some details from [8,9].

For L^2-potential $v(x) = \sum V(2k)e^{2\pi i k x}$ on $[0, 1]$, either real- or complex-valued, we consider the equation

$$-y'' + v(x)y = \lambda y, \quad x \in [0, 1],$$

with periodic boundary conditions

$$y(0) = y(1), \qquad y'(0) = y'(1), \tag{12}$$

or antiperiodic boundary conditions

$$y(0) = -y(1), \qquad y'(0) = -y'(1). \tag{13}$$

In either case, corresponding eigenfunctions f extended on $[1, 2]$ as $f(1 + x) = f(x)$ in the case (12), or as $f(1 + x) = -f(x)$ in the case (13), produce eigenfunctions (with the same eigenvalues as in (12), (13)) of the Hill's operator

$$-y'' + v(x)y = \lambda y, \quad 0 \leqslant x \leqslant 2, \tag{14}$$

with periodic boundary conditions

$$y(0) = y(2), \qquad y'(0) = y'(2).$$

Pairs $\{\lambda_n^-, \lambda_n^+\}$ of eigenvalues of (14) are close to $\pi^2 n^2$ for n large enough; if n is even (or odd) they come from (12) (or (13)). The pure Fourier method with

$$y(x) = \sum_{-\infty}^{\infty} f_p e^{\pi i p x}, \quad 0 \leqslant x \leqslant 2,$$

transforms (14) into the system

$$\left(\pi^2 p^2 - \lambda\right) f_p + \sum_{m \in \mathbb{Z}} V(p - m) f_m = 0, \quad p \in \mathbb{Z}, \tag{15}$$

where $V(j) = 0$ if j is odd. If we look for eigenvalues close to $\pi^2 n^2$ and put

$$\lambda = \pi^2 n^2 + z$$

then the system (15) could be split into a system of two scalar equations

$$\begin{aligned} &\text{(a)} \quad -zx + V(-2n)y + \left[S^n J\widehat{V}, F\right] = 0, \\ &\text{(b)} \quad V(2n)x - zy + \left[S^{-n} J\widehat{V}, F\right] = 0, \end{aligned} \tag{16}$$

where

$$x = f_{-n}, \qquad y = f_n, \qquad F = (f_k)_{k \in \mathbb{Z}(n)}, \qquad \mathbb{Z}(n) = \mathbb{Z} \setminus \{-n, n\},$$

and one vector equation in $\ell^2(\mathbb{Z}(n))$

$$x \cdot \left(S^n \widehat{V}\right) + y \cdot \left(S^{-n} \widehat{V}\right) + (A_n - z)F = 0, \tag{17}$$

where $\widehat{V} = V|_{\mathbb{Z}(n)}$,

$$S : \{g(k)\} \to \{g(k+1)\}, \qquad J : \{g(k)\} \to \{g(-k)\},$$

and $A_n : \ell^2(\mathbb{Z}(n)) \to \ell^2(\mathbb{Z}(n))$ is defined by the matrix

$$A_n(k, j) = \pi^2\bigl(k^2 - n^2\bigr)\delta_{kj} + V(k - j), \quad k, j \in \mathbb{Z}(n). \tag{18}$$

For n large enough the operator $(A_n - z)$ is invertible and (17) determines F in terms of x, y. If we substitute this F into (16) we come to two linear equations for x, y (see [8, (2.14), p. 624])

$$\begin{pmatrix} -z + \alpha(-n, z) & B^-(-n, z) \\ B^+(n, z) & -z + \alpha(n, z) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{19}$$

where

$$\alpha(n, z) = \bigl\langle S^{-n} JV, (z - A_n)^{-1}(S^{-n}V)_{\mathbb{Z}(n)}\bigr\rangle, \tag{20}$$

$$B^-(n, z) = V(-2n) + \beta(-n, z), \qquad B^+(n, z) = V(2n) + \beta(n, z) \tag{21}$$

with

$$\beta(n, z) = \bigl\langle S^{-n} JV, (z - A_n)^{-1}(S^{n}V)_{\mathbb{Z}(n)}\bigr\rangle. \tag{22}$$

A nonzero solution (x, y) for (19) exists if and only if z is a root of the determinant of this system. With

$$\alpha(-n, z) = \alpha(n, z) \tag{23}$$

(Lemma 2.2 in [8]) it means that for

$$\zeta = z - \alpha(n, z), \tag{24}$$

$$\zeta^2 - B^-(n, z)B^+(n, z) = 0. \tag{25}$$

Let $\|V\|$ be the norm of $(V(n))$ in ℓ^2, that is $\|V\| = (\sum |V(k)|^2)^{1/2}$. By Lemma 0.4 in [9] there exists an absolute constant $K \geqslant 1$ such that for $n > n_0 := 2K(1 + \|V\|)$

$$\bigl|z_n^\pm\bigr| \leqslant M = K(1 + \|V\|), \quad z_n^\pm = \lambda_n^\pm - \pi^2 n^2. \tag{26}$$

In [9], Lemma 1.5, it is proven that for every fixed $n \geqslant n_0$

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\alpha(n, z)\right| \leqslant \frac{4}{\pi^4 n^2}\|V\|^2 < 1/2.$$

Thus (24) defines in the disc $|z| < M$ a holomorphic mapping $\zeta(z) = z - \alpha(n, z)$ such that

$$1/2 < \left| \frac{\mathrm{d}}{\mathrm{d}z} \zeta(z) \right| < 3/2.$$

From here it follows

$$2^{-1}\left|z_n^+ - z_n^-\right| \leqslant \left|\zeta\left(z_n^+\right) - \zeta\left(z_n^-\right)\right| \leqslant 2\left|z_n^+ - z_n^-\right|,$$

so, taking into account that

$$\left|\lambda_n^+ - \lambda_n^-\right| = \left|z_n^+ - z_n^-\right|$$

we obtain

$$2^{-1}\left|\lambda_n^+ - \lambda_n^-\right| \leqslant \left|\zeta_n^+ - \zeta_n^-\right| \leqslant 2\left|\lambda_n^+ - \lambda_n^-\right|, \tag{27}$$

where $\zeta_n^+ = \zeta(z_n^+)$ and $\zeta_n^- = \zeta(z_n^-)$. Now from (25) it follows

$$\left|\zeta_n^\pm\right| = \left|B^-\left(n, z_n^\pm\right) B^+\left(n, z_n^\pm\right)\right|^{1/2} \leqslant \max\left\{\left|B^-\left(n, z_n^\pm\right)\right|, \left|B^+\left(n, z_n^\pm\right)\right|\right\}, \tag{28}$$

therefore

$$\left|\zeta_n^+ - \zeta_n^-\right| \leqslant \left|\zeta_n^+\right| + \left|\zeta_n^-\right| \leqslant 2\max\left\{\left|B^-\left(n, z_n^\pm\right)\right|, \left|B^+\left(n, z_n^\pm\right)\right|\right\}.$$

Thus by (27) we obtain

$$\left|\lambda_n^+ - \lambda_n^-\right| \leqslant 4\max\left\{\left|B^-\left(n, z_n^\pm\right)\right|, \left|B^+\left(n, z_n^\pm\right)\right|\right\}. \tag{29}$$

On the other hand, if μ_n denotes the Dirichlet eigenvalue of (1) on $[0, 1]$ that is close to $\pi^2 n^2$, then $|\mu_n - \lambda_n^+|$ and $|\mu_n - \lambda_n^-|$ can be estimated from above by a multiple of the right-hand side of (29)—this follows immediately from Theorem 3.5 and formula (2.28) in [9]. More precisely, let

$$d_n = \max\left\{\left|\lambda_n^+ - \lambda_n^-\right|, \left|\mu_n - \lambda_n^+\right|, \left|\mu_n - \lambda_n^-\right|\right\}, \tag{30}$$

where λ_n^+, λ_n^-, and μ_n are the eigenvalues of (1) that are close to $\pi^2 n^2$, respectively with periodic (for n even), antiperiodic (for n odd), and Dirichlet boundary conditions. Then the following statement holds.

Proposition 1. *If $n \geqslant n_0$ then*

$$d_n \leqslant C\max\left\{\left|B^-\left(n, z_n^\pm\right)\right|, \left|B^+\left(n, z_n^\pm\right)\right|\right\}, \tag{31}$$

where C is an absolute constant.

In order to use (31) and give estimates of d_n in terms of Fourier coefficients of V we need, in view of (21), an explicit form of $\beta(n,z)$.

Proposition 2. *For $n \geqslant n_0 = 2K(1+\|V\|)$ and $|z| \leqslant M = K(1+\|V\|)$ we have $\beta(n,z) = \sum_{k=1}^{\infty} \beta_k(n,z)$, where*

$$\beta_k(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)}{[z+\pi^2(n^2-j_1^2)]\cdots[z+\pi^2(n^2-j_k^2)]}. \tag{32}$$

Moreover, for $|z| \leqslant M$

$$|\beta(n,z)| \leqslant S(n) = \sum_{k=1}^{\infty} S_k(n), \tag{33}$$

where

$$S_k(n) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{|V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)|}{|n^2-j_1^2|\cdots|n^2-j_k^2|}. \tag{34}$$

Proof. By (22) we have

$$\beta(n,z) = \left\langle S^{-n}JV, (z-A_n)^{-1}(S^nV)_{\mathbb{Z}(n)}\right\rangle.$$

Further, as usual, we identify operators in ℓ^2 with corresponding matrices. Set

$$D_n = \pi^2\left(k^2-n^2\right)\delta_{kj}, \quad V = V(k-j);$$

the operator $z - D_n$ is invertible for $n \geqslant n_0$ and $|z| \leqslant M$, and we have by (18)

$$z - A_n = z - D_n - V = (I-T_n)(z-D_n),$$

where

$$T_n = V(z-D_n)^{-1}, \tag{35}$$

so

$$(z-A_n)^{-1} = (z-D_n)^{-1}(I-T_n)^{-1}.$$

By Lemma 2.1 in [8]

$$\|T_n\|_{\ell^2} \leqslant \mathrm{const}/|n| < 1/2 \quad \text{for } |n| \geqslant n_0,$$

therefore the inverse operator $(I - T_n)^{-1}$ exists, and moreover,

$$(I - T_n)^{-1} = \sum_{k=1}^{\infty} T_n^k, \quad |n| \geqslant n_0.$$

Thus

$$(z - A_n)^{-1} = \sum_{k=1}^{\infty} (z - D_n)^{-1} T_n^k,$$

so $\beta(n, z) = \sum_{k=1}^{\infty} \beta_k(n, z)$, where

$$\beta_k(n, z) = \bigl\langle S^{-n} J V, (z - D_n)^{-1} T_n^k (S^n V)_{\mathbb{Z}(n)} \bigr\rangle.$$

Now taking into account that

$$(z - D_n)^{-1} = \frac{1}{z - \pi^2(k^2 - n^2)} \delta_{kj} \tag{36}$$

we obtain (32).

On the other hand, for $n \geqslant n_0$ and $|z| \leqslant M$ we have

$$\bigl|z + \pi^2\bigl(n^2 - k^2\bigr)\bigr| \geqslant \pi^2\bigl|n^2 - k^2\bigr| - |z| \geqslant \bigl|n^2 - k^2\bigr|, \quad k \neq \pm n,$$

thus (32) implies (33). □

If v is a real-valued potential we have $V(-2n) = \overline{V(2n)}$; moreover $\beta(-n, z) = \overline{\beta(n, \bar{z})}$ (see Lemma 3 in [2]), thus by (21) $B^-(n, z) = \overline{B^+(n, \bar{z})}$. These additional properties imply that the quasi-quadratic equation (25) could be split in two quasi-linear equations. This observation leads to the following

Proposition 3. *For $n \geqslant n_0 = 2K(1 + \|V\|)$ there exists a sequence (z_n) with $|z_n| \leqslant M \leqslant K(1 + \|V\|)$ such that if*

$$\eta_n = V(2n) + \beta(n, z_n), \tag{37}$$

then

$$\left(\frac{1}{2} - \delta_n\right)\gamma_n \leqslant |\eta_n| \leqslant \left(\frac{1}{2} + \delta_n\right)\gamma_n, \quad \gamma_n = \lambda_n^+ - \lambda_n^-, \tag{38}$$

with $\delta_n = \|V\|^2/4n^2 \to 0$.

Proof. For the proof see [2], Theorem 8, where (37) becomes the main tool and object to analyze. □

We will need the following elementary statement.

Lemma 4.

$$\sum_{j \neq \pm n} \frac{1}{|n^2 - j^2|} \leqslant 4 \frac{1 + \log(2n)}{2n}. \tag{39}$$

3. Superexponential weights

Consider a weight sequence

$$\omega(j) = \exp\bigl(a|j|^b\bigr), \quad a > 0,\ b > 1,\ j \in \mathbb{Z}. \tag{40}$$

Consider also a family of weights

$$\Omega_c(m) = \exp\bigl(cm(\log m)^{(b-1)/b}\bigr), \quad c > 0,\ b > 1,\ m \in \mathbb{N}. \tag{41}$$

Theorem 5. *Suppose $v(x) = \sum_n V(2n)\exp(2n\pi \mathrm{i}x)$ is an 1-periodic potential such that*

$$\|V\|_\omega = \sup |V(n)|\omega(n) < \infty, \tag{42}$$

where $\omega(n)$ is the weight (40). *Then*

$$\sup_n d_n\, \Omega_c(2n) < \infty, \quad \forall c < c^*, \tag{43}$$

where

$$c^* = a^{1/b} b \left(\frac{1}{(b-1)}\right)^{(b-1)/b}. \tag{44}$$

Proof. We begin with some preliminary inequalities. Observe that by Hölder inequality

$$|x_1 + \cdots + x_k| \leqslant \bigl(|x_1|^b + \cdots + |x_k|^b\bigr)^{1/b} k^{(b-1)/b},$$

so

$$k^{1-b}|x_1 + \cdots + x_k|^b \leqslant |x_1|^b + \cdots + |x_k|^b.$$

Hence

$$\exp\bigl(ak^{1-b}|x_1 + \cdots + x_k|^b\bigr) \leqslant \omega(x_1)\cdots\omega(x_k). \tag{45}$$

Lemma 6. *Suppose $a > 0, b > 1$; then for every $\mu > 0$*

$$m^{\mu k} \exp\bigl(ak^{1-b}m^b\bigr) \geqslant \exp\bigl(c_\mu m(\log m)^{b-1)/b}\bigr), \quad \forall k, m \in \mathbb{N}, \tag{46}$$

where

$$c_\mu = a^{1/b} b[\mu/(b-1)]^{(b-1)/b}. \tag{47}$$

Indeed, consider the function

$$\varphi(x) = \mu x \log m + ax^{1-b}m^b, \quad x \geqslant 1.$$

Since

$$\varphi'(x) = \mu \log m - a(b-1)x^{-b}m^b$$

we have $\varphi'(x_0) = 0$ for $x_0 = [a(b-1)\mu^{-1}]^{1/b} m(\log m)^{-1/b}$, and $\varphi'(x) < 0$ for $x < x_0$, $\varphi'(x) > 0$ for $x > x_0$. Thus

$$\varphi(x) \geqslant \varphi(x_0) = c_\mu m(\log m)^{(b-1)/b},$$

which proves Lemma 6.

Now we continue the proof of Theorem 5. By (31) in Proposition 1 the theorem will be proven if we show that

$$\sup_{n\geqslant n_0} |B^+(n,z)|\Omega_c(2n) < \infty, \qquad \sup_{n\geqslant n_0} |B^-(n,z)|\Omega_c(2n) < \infty. \tag{48}$$

Obviously

$$\Omega_c(m)/\omega(m) \to 0 \quad \text{as } m \to \infty\ \forall c > 0$$

therefore we have for every $c > 0$

$$\sup_{n\geqslant n_0} |V(2n)|\Omega_c(2n) \leqslant \mathrm{const} \cdot \|V\|_\omega < \infty.$$

So, in view of (21), it is enough to prove that

$$\sup_{|n|\geqslant n_0} \bigl|\beta\bigl(n, z_n^\pm\bigr)\bigr|\Omega_c(2n) < \infty. \tag{49}$$

On the other hand, by Proposition 2

$$\bigl|\beta\bigl(n, z_n^\pm\bigr)\bigr| \leqslant S(n) = \sum_{k=1}^{\infty} S_k(n),$$

where $S_k(n)$ is defined by (34). By (46) (with $k+1$ instead of k and $x_1 = -n + j_1, x_2 = j_2 - j_1, \dots, x_k = j_k - j_{k-1}, x_{k+1} = n - j_k$) we obtain

$$\exp\left(a(k+1)\left(\frac{2n}{k+1}\right)^b\right) \leqslant \omega(n+j_1)\omega(j_2-j_1)\cdots\omega(j_k-j_{k-1})\omega(n-j_k),$$

therefore

$$\sigma_k(n) := |S_k(n)| \exp\left(a(k+1)\left(\frac{2n}{k+1}\right)^b\right) \leqslant \sum_{j_1,\dots,j_k \neq \pm n} \frac{\alpha(n+j_1)\alpha(j_2-j_1)\cdots\alpha(j_k-j_{k-1})\alpha(n-j_k)}{|n^2-j_1^2||n^2-j_2^2|\cdots|n^2-j_k^2|},$$

where

$$\alpha(s) = |V(s)|\omega(s) \leqslant \|V\|_\omega, \quad s \in \mathbb{Z}.$$

From here it follows

$$\sigma_k(n) \leqslant \|V\|_\omega^{k+1} \sum_{j_1,\dots,j_k \neq \pm n} \frac{1}{(n^2-j_1^2)(n^2-j_2^2)\cdots(n^2-j_k^2)},$$

thus by Lemma 4

$$\sigma_k(n) \leqslant \|V\|_\omega^{k+1}\left(\sum_{j \neq \pm n} \frac{1}{|n^2-j^2|}\right)^k \leqslant \|V\|_\omega^{k+1}\left(4\frac{1+\log(2n)}{2n}\right)^k. \tag{50}$$

Fix $c \in (0, c^*)$ and choose $\tilde{c}$ so that $c < \tilde{c} < c^*$. Then obviously

$$2n\Omega_c(2n) \leqslant A\Omega_{\tilde{c}}(2n)$$

for some constant $A = A(c, \tilde{c}) < \infty$. Choose a $\mu \in (0,1)$ so that

$$\tilde{c} = c_\mu = a^{1/b} b[\mu/(b-1)]^{(b-1)/b}$$

(see (47) in Lemma 6). Then by (46) in Lemma 6 and (50) we have

$$\sum_{k=1}^\infty |S_k(n)|\Omega_c(2n) \leqslant A \sum_{k=1}^\infty |S_k(n)| \exp\left(a(k+1)\left(\frac{2n}{k+1}\right)^b\right)(2n)^{\mu(k+1)-1}$$
$$\leqslant A \sum_{k=1}^\infty \sigma_k(n)(2n)^{\mu k} \leqslant A \sum_{k=1}^\infty \|V\|_\omega^{k+1}\left(4\frac{1+\log(2n)}{(2n)^{(1-\mu)}}\right)^k.$$

Choose n_μ so that for $n \geqslant n_\mu$

$$4\|V\|_\omega \frac{1+\log(2n)}{(2n)^{(1-\mu)}} \leqslant \frac{1}{2}; \tag{51}$$

then by (51) we obtain

$$\sup_{n\geqslant n_\mu} \sum_{k=1}^{\infty} |S_k(n)|\Omega_c(2n) \leqslant A\|V\|_\omega \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

It completes the proof of Theorem 5. □

4. Real valued potentials

If the potential v is real-valued then the Dirichlet eigenvalue μ_n is situated between λ_n^- and λ_n^+ (see [10]), therefore the diameters of spectral triangles coincides with spectral gaps

$$\gamma_n = \lambda_n^+ - \lambda_n^-.$$

By Theorem 5 if

$$|V(m)| \leqslant \exp\bigl(-\mathrm{const}\cdot |m|^b\bigr), \quad b > 1, \tag{52}$$

then

$$\gamma_n \leqslant \exp\bigl(-\mathrm{const}\cdot 2n(\log 2n)^{(b-1)/b}\bigr). \tag{53}$$

In this section we explain that the estimate (53) is "sharp" in the sense that there exists a real valued potential v with (52) such that its spectral gap sequence satisfies for infinitely many n an inequality

$$\gamma_n > \exp\bigl(-\mathrm{const}\cdot 2n(\log 2n)^{(b-1)/b}\bigr). \tag{54}$$

Construction. Let $a > 0$ and $b > 1$. Choose a sequence of *even* integers

$$0 < t_1 < t_2 < \cdots < t_\nu < \cdots \tag{55}$$

such that

$$t_{\nu+1} > 4\exp\bigl(at_\nu^b\bigr), \quad \nu = 1, 2, \ldots. \tag{56}$$

Set for each $\nu = 1, 2, \ldots,$

$$k_\nu := \Bigl[t_\nu^{-1} \mathrm{e}^{at_\nu^b}\Bigr], \quad 2n_\nu := (k_\nu + 1)t_\nu, \quad \nu = 1, 2, \ldots, \tag{57}$$

where $[x]$ means the integer part of x. Observe that

$$k_\nu t_\nu \leqslant \exp\bigl(a t_\nu^b\bigr) \leqslant 2n_\nu \leqslant \exp\bigl(a t_\nu^b\bigr) + t_\nu, \tag{58}$$

$$t_\nu < \left(\frac{\log 2n_\nu}{a}\right)^{1/b} \tag{59}$$

and

$$t_{\nu+1} > 4n_\nu. \tag{60}$$

Set

$$V(j) = \begin{cases} \exp\bigl(-a t_\nu^b\bigr) & \text{if } j = \pm t_\nu, \\ 0 & \text{if } j \neq \pm t_\nu, \end{cases} \qquad v(x) = \sum_{j \in \mathbb{Z}} V(j) \exp(\pi \mathrm{i} j x). \tag{61}$$

Theorem 7. *If v is the potential given by* (61) *and (γ_n) is the corresponding spectral gap sequence then for any constant $c > 3a^{1/b}$ there exists ν_0 such that*

$$\gamma_{n_\nu} > \exp\bigl(-c(2n_\nu)(\log 2n_\nu)^{(b-1)/b}\bigr), \quad \nu \geqslant \nu_0. \tag{62}$$

Proof. For convenience the proof is divided into several steps.

Step 1. Fix $n = n_\nu$. From (60) and (61) it follows that $V(2n) = 0$, therefore, in view of Proposition 3 we have

$$\gamma_n \geqslant |\beta(n, z_n)|, \quad n > n_0, \ |z_n| \leqslant M.$$

By Proposition 2 $\beta(n,z) = \sum_{k=1}^\infty \beta_k(n,z)$, where

$$\beta_k(n,z) = \sum_{j_1,\ldots,j_k \neq \pm n} \frac{V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)}{[z+\pi^2(n^2-j_1^2)]\cdots[z+\pi^2(n^2-j_k^2)]}.$$

Due to (61) a k-tuple $j = (j_1, \ldots, j_k)$ yields a nonzero product

$$V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k) \neq 0$$

if and only if the numbers

$$\begin{aligned} &x_1 := n + j_1, \qquad x_2 := j_2 - j_1, \qquad \ldots, \\ &x_k := j_k - j_{k-1}, \qquad x_{k+1} := n - j_k \end{aligned} \tag{63}$$

belong to the set $\{\pm t_1, \pm t_2, \ldots\}$. We call such a k-tuple j "*admissible*" and denote the set of all admissible k-tuples by $J_k(n)$. In the sequel it would be convenient to regard each admissible k-tuple as a "walk" on the real line from $-n$ to n with $k+1$ steps $x_1, \ldots, x_{k+1}$; then j_m, $m = 1, \ldots, k$, is the point reached after m steps.

Observe, that if j is an admissible k-tuple then by (63) the corresponding steps $x_m \in \{\pm t_1, \pm t_2, \dots\}$ satisfy the equation

$$x_1 + \cdots + x_{k+1} = 2n. \tag{64}$$

Now we show (by induction in k) that if $k < k_\nu$ then (64) has no solution $(x_1, \dots, x_{k+1})$ with $x_m \in \{\pm t_1, \pm t_2, \dots\}$. If $k = 1$ the claim follows from (58) and (60). Let the claim holds for $k < k' < k_\nu$.

Assume that (64) has a solution with $k = k'$. If $x_m \leqslant t_\nu$ for all m then $\sum_m x_m \leqslant k t_\nu < 2n$ because $2n = (k_\nu + 1)t_\nu$. So, the solution should contain some "huge" steps $x_m > t_\nu$. Let $x_{\tilde m} = t_{\tilde\nu}$, $\tilde\nu > \nu$, be one of the largest positive steps of the solution. In view of (60), if every negative step has length $< x_{\tilde m}$ then

$$\sum_1^{k+1} x_m \geqslant t_{\tilde\nu} - \sum_{|x_m| < x_{\tilde m}} |x_m| \geqslant t_{\tilde\nu} - k_\nu t_{\tilde\nu - 1} > 2n_{\tilde\nu - 1} \geqslant 2n,$$

so by (64) there are some large negative steps with length $\geqslant x_{\tilde m}$. If there is a negative step which length equals $x_{\tilde m}$ one may reduce the number of steps and get a solution of (64) when $k = k' - 2$, which contradicts the choice of k'. But it is impossible to have a case where there is a negative step with length $t_{\nu_1} > x_{\tilde m} = t_{\tilde\nu}$ because then the sum of positive steps will be less than $k_\nu t_{\tilde\nu} < 2n_{\tilde\nu} < t_{\nu_1}$, so (64) fails for $k = k'$.

Since only the admissible k-tuples yield nonzero terms of $\beta_k(n, z_n)$ the above observation shows that

$$\beta_k(n, z_n) = 0 \quad \text{for } k < k_\nu,$$

therefore

$$\beta(n, z_n) = \sum_{k \geqslant k_\nu} \beta_k(n, z_n).$$

Moreover, the same argument shows that $\beta_{k_\nu+1} = 0$ and $\beta_{k_\nu+3} = 0$ because the equation

$$x_1 + \cdots + x_k = 2n = (k_\nu + 1)t_\nu$$

has no solution with $x_1, \dots, x_k \in \{\pm t_1, \pm t_2, \dots\}$ if $k = k_\nu + 1$ or $k = k_\nu + 3$. Thus

$$\beta(n, z_n) = \beta_{k_\nu}(n, z_n) + \beta_{k_\nu+2}(n, z_n) + \sum_{k \geqslant k_\nu+4} \beta_k(n, z_n). \tag{65}$$

Since $v(x)$ is a real-valued potential the numbers $\{z_n\}$ are real, therefore the terms $\beta_k(n, z_n)$ are real-valued as well.

The idea of the proof is that $\beta_{k_\nu}(n, z_n) > 0$ gives the main part of $\beta(n, z_n)$. In Step 2 we estimate $\beta_{k_\nu}(n, z_n)$ from below; the obtained lower bound would prove the theorem provided it would be shown that the impact of other terms of (65) is negligible. This is done in Steps 3–10 below; we prove that

$$\beta_{k_\nu}(n, z_n) \gg \sum_{k \geqslant k_\nu + 4} |\beta_k(n, z_n)|.$$

For a technical reason we consider $\beta_{k_\nu+2}(n, z_n)$ separately; it is proven in Step 5 that

$$\beta_{k_\nu}(n, z_n) \gg -\beta_{k_\nu+2}(n, z_n).$$

Step 2. There exist constants ν_0 and $C_0 > 0$ such that for $\nu \geqslant \nu_0$ we have

$$|\beta_{k_\nu}(n, z_n)| > \frac{C_0}{\pi^{2k_\nu}} S_{k_\nu}, \tag{66}$$

where

$$S_{k_\nu} = \frac{V(t_\nu)^{k_\nu}}{(k_\nu!)^2 t_\nu^{2k_\nu}}. \tag{67}$$

Indeed, it is easy to see (using the same argument as in Step 1) that there is only one admissible k_ν-tuple, namely

$$j_1 = -n + t_\nu, \qquad j_2 = -n + 2t_\nu, \qquad \dots, \qquad j_{k_\nu} = -n + k_\nu t_\nu,$$

so the sum $\beta_{k_\nu}(n, z_n)$ has only one term:

$$\beta_{k_\nu}(n, z_n) = \frac{V(t_\nu)^{k_\nu}}{[z_n + \pi^2(n^2 - j_1^2)] \cdots [z_n + \pi^2(n^2 - j_{k_\nu}^2)]}. \tag{68}$$

Let n_0 and M are the constants from Proposition 3; then $|z_n| \leqslant M$ for $n \geqslant n_0$. Choose ν_0 so that $n_{\nu_0} \geqslant n_0$; then for $n = n_\nu$ with $\nu \geqslant \nu_0$ we have $|z_n| \leqslant M$, thus for every $m = 1, \dots, k_\nu$

$$\left|z_n + \pi^2\left(n^2 - j_m^2\right)\right| \leqslant M + \pi^2\left(n^2 - j_m^2\right) \leqslant \pi^2\left(n^2 - j_m^2\right)(1 + M/n).$$

Since $(1 + M/n)^{k_\nu} \leqslant (1 + M/n)^n \leqslant \mathrm{e}^M$ we obtain that

$$\left|z_n + \pi^2\left(n^2 - j_1^2\right)\right| \cdots \left|z_n + \pi^2\left(n^2 - j_{k_\nu}^2\right)\right| \leqslant \mathrm{e}^M \pi^{2k_\nu}\left(n^2 - j_1^2\right) \cdots \left(n^2 - j_{k_\nu}^2\right).$$

On the other hand, from $2n = (k_\nu + 1)t_\nu$ it follows

$$n^2 - j_m^2 = n^2 - (-n + mt_\nu)^2 = mt_\nu \cdot (2n - mt_\nu) = mt_\nu(k_\nu + 1 - m)t_\nu,$$

therefore

$$\left(n^2 - j_1^2\right) \cdots \left(n^2 - j_{k_\nu}^2\right) = (k_\nu!)^2 t_\nu^{2k_\nu}.$$

Hence by (68) we obtain (66) with $C_0 = \mathrm{e}^{-M}$.

Let us estimate S_{k_ν} from below. By (58) and (61) we have $V(t_\nu) \geqslant 1/(k_\nu t_\nu) > 1/2n$ and $k_\nu \leqslant 2n$; therefore (67), together with $k_\nu! < {k_\nu}^{k_\nu}$, implies

$$S_{k_\nu} > \frac{(2n)^{-k_\nu}}{(k_\nu t_\nu)^{2k_\nu}} > \frac{1}{(2n)^{3k_\nu}} \tag{69}$$

and as well

$$S_{k_\nu} > \frac{(k_\nu t_\nu)^{-k_\nu}}{(k_\nu t_\nu)^{2k_\nu}} \geqslant \left(\mathrm{e}^{-at_\nu^b}\right)^{3k_\nu} = \mathrm{e}^{-3at_\nu^b k_\nu},$$

so by (58) and (59)

$$S_{k_\nu} > \exp\bigl(-3a^{1/b}(2n)(\log 2n)^{(b-1)/b}\bigr). \tag{70}$$

Step 3.

$$\sum_{k>5k_\nu} |\beta_k(n, z_n)| < \frac{1}{(2n)^{k_\nu}} S_{k_\nu}. \tag{71}$$

Indeed, by Proposition 2 for $n \geqslant n_0$ $|\beta_k(n, z_n)| \leqslant S_k(n)$, where

$$S_k(n) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{|V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)|}{|n^2-j_1^2|\cdots|n^2-j_k^2|}.$$

Since $V(\cdot) \leqslant 1$ we have by Lemma 4

$$S_k(n) \leqslant \left(\sum_{i\neq\pm n} \frac{1}{|n^2-i^2|}\right)^k \leqslant \left(\frac{4+4\log 2n}{2n}\right)^k.$$

Therefore by (69)

$$S_k(n) \leqslant \left(\frac{4+4\log 2n}{(2n)^{1/5}}\right)^k \frac{1}{(2n)^{4k_\nu}} \leqslant 2^{-k} \frac{1}{(2n)^{k_\nu}} S_{k_\nu}$$

thus for large enough n we obtain

$$\sum_{k>5k_\nu} S_k(n) \leqslant \frac{1}{(2n)^{k_\nu}} S_{k_\nu}.$$

Step 4. In Proposition 2 we estimated $|\beta_k(n, z_n)|$ from above by sums $S_k(n)$ given by (34). Here we need a more precise estimate: If $M = K(1+\|V\|)$ is the constant from Proposition 2 and ν_0 is chosen (as in Step 2) so that $n = n_\nu > n_0$ for $\nu \geqslant \nu_0$, then for $n = n_\nu$ with $\nu \geqslant \nu_0$ we have

$$|\beta_k(n,z)| \leqslant \frac{C_1}{\pi^{2k}} S_k(n), \quad k \leqslant 5n, \tag{72}$$

where $C_1 >$ is a constant and

$$S_k(n) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{|V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)|}{|n^2-j_1^2|\cdots|n^2-j_k^2|}.$$

Indeed, by Proposition 2

$$\beta_k(n,z_n) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)}{[z_n+\pi^2(n^2-j_1^2)]\cdots[z_n+\pi^2(n^2-j_k^2)]}.$$

By the choice of ν_0 we have $|z_n| \leqslant M$ for $\nu \geqslant \nu_0$, therefore (since $|n^2-j_m^2| > 4n-1$)

$$\left|z_n+\pi^2\left(n^2-j_m^2\right)\right| \geqslant \pi^2\left|n^2-j_m^2\right| - M \geqslant \pi^2\left|n^2-j_m^2\right|(1-M/(10n))$$

for every $m = 1,\dots,k$. Since $(1-M/(10n))^{5n} \to \mathrm{e}^{-M/2}$ there is a constant C_1 (one may take $C_1 = \mathrm{e}^M$) such that $(1-M/(10n))^{-5n} \leqslant C_1$. Thus for $k \leqslant 5k_\nu < 5n$

$$(1-M/(10n))^{-k} \leqslant (1-M/(10n))^{-5n} \leqslant C_1,$$

and we obtain

$$\prod_{m=1}^{k}\left|z_n+\pi^2\left(n^2-j_m^2\right)\right|^{-1} \leqslant \frac{C_1}{\pi^{2k}} \prod_{m=1}^{k}\left|n^2-j_m^2\right|^{-1},$$

which immediately implies (72).

Step 5. Recall that $n = n_\nu$ is fixed. We say that an admissible k-tuple $j = (j_1,\dots,j_k)$ is *essential* if $j_1,\dots,j_k \neq \pm n$ and there exist

$$x_1,\dots,x_{k+1} \in \{\pm t_1,\dots,\pm t_\nu\} \tag{73}$$

such that (64) holds and

$$j_1 = -n + x_1, \qquad j_2 = j_1 + x_2, \qquad \dots, \qquad j_k = j_{k-1} + x_k. \tag{74}$$

In other words, a walk j is essential if the corresponding steps are not larger than t_ν.

Let J_k' be the set of all essential k-tuples, and let J_k'' be its complement in the set of all admissible k-tuples $J_k(n)$. Then we may split the sum $S_k(n)$, $k > k_\nu$, into two

$$S_k(n) = S_k'(n) + S_k''(n),$$

where $S_k'(n)$ and $S_k''(n)$ are the subsums corresponding respectively to essential k-tuples $j \in J_k'$ and "nonessential" admissible k-tuples $j \in J_k''$. Next we show that the impact of the sums of "nonessential" terms is very small.

Lemma 8.

$$\sum_{k>k_\nu} \bigl|S_k''(n_\nu)\bigr| \leqslant \exp\bigl(-a(4n_\nu)^b\bigr), \quad \nu > \nu_0. \tag{75}$$

Proof. Fix $k > k_\nu$. If $j = (j_1, \dots, j_k) \in J_k''$ then at least one of the corresponding steps x_m (by absolute value) is strictly greater or equal to $t_{\nu+1}$. Taking into account (61), and then (60), we obtain with $n = n_\nu$

$$V(n + j_1)V(j_2 - j_1)\cdots V(j_k - j_{k-1})V(n - j_k) < V(t_{\nu+1}) < V(4n),$$

thus

$$\bigl|S_k''(n)\bigr| \leqslant \sum_{J_k''} \frac{V(4n)}{|n^2 - j_1^2|\cdots|n^2 - j_k^2|} \leqslant \Biggl(\sum_{j\neq\pm n} \frac{1}{|n^2 - j^2|}\Biggr)^k \exp\bigl(-a\cdot(4n)^b\bigr).$$

In view of Lemma 4

$$\sum_{j\neq\pm n} \frac{1}{|n^2 - j^2|} < 4\frac{1+\log 2n}{2n} \leqslant \frac{1}{2} \quad \text{for } n \geqslant 32,$$

so there exists ν_0 such that

$$\sum_{j\neq\pm n} \frac{1}{|n^2 - j^2|} \leqslant \frac{1}{2} \quad \text{for } \nu > \nu_0.$$

Hence (75) holds. □

Step 6. Here we estimate only the sum of the negative terms of $\beta_{k_\nu+2}(n, z_n)$. We are not interested in positive terms of $\beta_{k_\nu+2}(n, z_n)$ because our goal is to estimate $\beta_{k_\nu}(n, z_n) + \beta_{k_\nu+2}(n, z_n)$.

Claim.

$$-\beta_{k_\nu+2}(n, z_n) < \frac{1+\log 2n}{8n} S_{k_\nu}. \tag{76}$$

Recall that if we regard every $j \in J'_{k_\nu+2}$ as a walk from $-n$ to n with steps $x_1, \dots, x_{k_\nu+3}$ then $j \in J'_{k_\nu+2}$ if and only if

$$x_1 + \cdots + x_{k_\nu+3} = 2n = (k_\nu + 1)t_\nu, \quad x_i \in \{\pm t_1, \dots, \pm t_\nu\}. \tag{77}$$

It is easy to see by using the same argument that was used in Step 1 that $x_1, \dots, x_{k_\nu+3}$ satisfy (77) if and only if for some indices $i_1, i_2 \leqslant k_\nu + 3$ and $s \leqslant \nu$ we have

$$x_{i_1} = -t_s, \qquad x_{i_2} = t_s, \qquad x_i = t_\nu \quad \text{for } i \neq i_1, i_2. \tag{78}$$

Let σ be the subsum of negative terms in $\beta_{k_\nu+2}(n, z_n)$. Since $V(\cdot) \geqslant 0$ the term that corresponds to a given $j \in J'_{k_\nu+2}$ is positive if $j_1, \ldots, j_{k_\nu+2} \in (-n, n)$ because

$$P(j) = \left(n^2 - j_1^2\right) \cdots \left(n^2 - j_{k_\nu+2}^2\right)$$

is positive. Thus a $j \in J'_{k_\nu+2}$ would yield a negative term only if either $j_1 < -n$ or $j_k > n$ (nothing else is possible due to (78)). In other words, either the first step of j should be negative (equal to $-t_s$), or the last step should be negative. Let σ^* (respectively σ^{**}) be the subsum of σ related to j with negative first step (respectively negative last step). Since

$$j = (j_1, \ldots, j_k) \in J' \quad \Leftrightarrow \quad \bar{j} := (-j_k, -j_{k-1}, \ldots, -j_1) \in J'_k$$

and $P(j) = P(\bar{j})$, we have $\sigma^* = \sigma^{**}$. Thus $\sigma = 2\sigma^*$, so it is enough to estimate only σ^*.

Observe that if $J_* \subset J'_{k_\nu+2}$ is the subset of walks that generates σ^* then

$$j \in J_* \quad \Leftrightarrow \quad \exists i_2, s\colon \quad x_1 = -t_s, \quad x_{i_2} = t_s, \quad x_i = t_\nu \quad \text{for } i \neq 1, i_2. \tag{79}$$

Consider the map

$$\tau(j) : J_* \to \mathbb{Z}, \qquad b(j) = j_{i_2-1};$$

from (79) it follows immediately that the map τ is injective. By (79) each term of σ^* has the form

$$\frac{[V(t_\nu)]^{k_\nu}[V(t_s)]^2}{(n^2 - j_1^2)(n^2 - j_2^2) \cdots (n^2 - j_{k_\nu+2}^2)}, \quad 1 \leqslant s \leqslant \nu - 1, \ j_1 = -n - t_s.$$

From

$$n^2 - j_1^2 = n^2 - (-n - t_s)^2 = -t_s(2n + t_s)$$

it follows $|n^2 - j_1^2| > 4n$. Therefore, since $V(t_s) \leqslant 1$, we obtain

$$|\sigma^*| \leqslant \frac{[V(t_\nu)]^{k_\nu}}{4n} \Pi_*^{-1} \sum_{j \in J^*} \frac{1}{n^2 - \tau^2(j)},$$

where

$$\Pi_* = \min_{j \in J_*} \prod_{i \neq 1, i_2} \left(n^2 - j_i^2\right).$$

Taking into account that $\tau(j)$ is an injection we obtain by Lemma 4 that

$$\sum_{j\in J^*} \frac{1}{n^2-\tau^2(j)} \leqslant \sum_{\tau\neq\pm n} \frac{1}{|n^2-\tau^2|} \leqslant \frac{4+4\log 2n}{2n}.$$

It remains to estimate Π_*. By (79) either $j_i = -n+(j-1)t_\nu - t_s,\ i < i_2-1$, or $j_i = -n+(j-2)t_\nu,\ i \geqslant i_2$. In both cases each of the intervals

$$(a_m, a_{m+1}], \quad a_m = -n + mt_\nu, \quad m = 1,\ldots,k_\nu - 1,$$

contains exactly one of the points j_i, $i \neq 1, i_2 - 1$.

Set

$$m^* = \max\{m:\ a_m \leqslant 0\}.$$

Notice that

(i) if $m < m^* - 1$ and $a_m < j_i \leqslant a_{m+1} \leqslant a_{m^*} \leqslant 0$ then $n^2 - j_i^2 > n^2 - a_m^2$;
(ii) if $m \geqslant m^*$ and $a_m < j_i \leqslant a_{m+1}$ then $j_i > 0$, thus $n^2 - j_i^2 \geqslant n^2 - a_{m+1}^2$.

Now from (i) and (ii) it follows

$$\prod_{i\neq 1,2,i_2} \bigl(n^2 - j_i^2\bigr) \geqslant \prod_{m=1}^{m^*-1} \bigl(n^2 - a_m^2\bigr) \prod_{m=m^*+1}^{k_\nu} \bigl(n^2 - a_m^2\bigr).$$

Taking into account that $n^2 - j_2^2 = (t_\nu - t_s)(2n - t_\nu - t_s) > 4n$, and

$$\prod_{m=1}^{k_\nu} \bigl(n^2 - a_m^2\bigr) = (k_\nu!)^2 {t_\nu}^{2k_\nu}$$

we obtain

$$\Pi_* \geqslant 4n(k_\nu!)^2 {t_\nu}^{2k_\nu} \frac{1}{n^2} = \frac{4}{n}(k_\nu!)^2 {t_\nu}^{2k_\nu}.$$

Hence

$$-\beta_{k_\nu+2}(n, z_n) \leqslant |\sigma^*| \leqslant \frac{1}{4}\frac{1+\log 2n}{2n} S_{k_\nu}.$$

Step 7. Fix k, $k_\nu + 4 \leqslant k \leqslant 5k_\nu$; then

$$S_k'(n) \leqslant W_k \cdot Z_k, \tag{80}$$

where

$$W_k = \max_{j \in J_k'} V(n + j_1)V(j_2 - j_1) \cdots V(j_k - j_{k-1})V(n - j_k), \tag{81}$$

$$Z_k = \sum_{j \in J_k'} \frac{1}{|n^2 - j_1^2| \cdots |n^2 - j_k^2|}. \tag{82}$$

In this step we estimate W_k. For every $j \in J_k'$ let $k'(j)$ denote the number of maximal steps ($= t_\nu$) of the walk j, and let

$$k' = k'(k) = \min\{k'(j),\ j \in J_k'\}. \tag{83}$$

Fix $j \in J_k'$ and regard j as a walk from $-n$ to n. Since there are $k'(j)$ steps with length t_ν and $k - k'(j)$ steps $\leqslant t_{\nu-1}$, we have

$$2n = (k_\nu + 1)t_\nu \leqslant k'(j)t_\nu + (k - k'(j))t_{\nu-1}.$$

From here it follows

$$k'(j)(t_\nu - t_{\nu-1}) > k_\nu t_\nu - k t_{\nu-1} = k_\nu(t_\nu - t_{\nu-1}) - (k - k_\nu)t_{\nu-1},$$

thus

$$k'(j) > k_\nu - (k - k_\nu)\frac{t_{\nu-1}}{t_\nu - t_{\nu-1}}.$$

Hence we obtain

$$k' \geqslant k_\nu - [(k - k_\nu)\varepsilon_\nu], \quad \varepsilon_\nu := t_{\nu-1}/(t_\nu - t_{\nu-1}). \tag{84}$$

Now we estimate W_k from above:

$$W_k \leqslant [V(t_\nu)]^{k'} \leqslant [V(t_\nu)]^{k_\nu - [(k-k_\nu)\varepsilon_\nu]} \leqslant [V(t_\nu)]^{k_\nu}(2n)^{[(k-k_\nu)\varepsilon_\nu]}; \tag{85}$$

we use (58) and (59) to explain that $(V(t_\nu))^{-1} = \mathrm{e}^{a t_\nu^b} \leqslant 2n$.

Step 8. Next we estimate Z_k. By (83) for each $j \in J_k'$ there are at least k', $k' > k_\nu - (k - k_\nu)\varepsilon_\nu$, steps equal to t_ν. Let

$$\tilde{k} = k_\nu - [2(k - k_\nu)\varepsilon_\nu]. \tag{86}$$

In order to estimate Z_k we show that for every $j = (j_m) \in J_k'$ at least $\tilde{k}$ of the points j_m with $x_{m+1} = t_\nu$ are "almost uniformly distributed" in $[-n, n]$.

Set

$$a_i = -n + i t_\nu, \quad \Delta_i = (a_i, a_{i+1}], \quad i = 1, \ldots, k_\nu - 1,$$

and

$$\tilde{\Delta}_i = (a_i - t_\nu/2, a_{i+1}], \quad i = 1, \dots, k_\nu - 1.$$

Proposition 9. *There exists a map*

$$\sigma : J'_k \to \mathbb{Z}_+^{\tilde{k}} \tag{87}$$

that maps every essential k-tuple $j = (j_m) \in J'_k$ into a $\tilde{k}$-subtuple $\sigma = \sigma(j)$ such that

(i) $\sigma_s = j_{m_s}$ *for some* m_s *with* $x_{m_s+1} = t_\nu$.
(ii) $\sigma_s \in \tilde{\Delta}_{i(s)}$, $1 \leqslant i(1) < i(2) < \cdots < i(\tilde{k}) \leqslant k_\nu - 1$.
(iii) *Let $\tau(j)$ be the complementary to $\sigma(j)$ subtuple of j. Then the map*

$$\tau : J'_k \to \mathbb{Z}_+^{k-\tilde{k}} \tag{88}$$

is injective.

Proof. We define the map σ by the following construction.

Construction. Fix $j \in J'_k$; the corresponding $\sigma(j) = (\sigma_1, \dots, \sigma_{\tilde{k}})$ is build by induction as follows.

Let $\tilde{\Delta}_{i(1)}$ be the first interval that contains points j_m with $x_{m+1} = t_\nu$. Set

$$\sigma_1 = j_{m_1} = \min\{j_m \in \tilde{\Delta}_{i(1)} \colon x_{m+1} = t_\nu\}. \tag{89}$$

Suppose $\sigma_1, \dots, \sigma_s$, $s < \tilde{k}$, are chosen so that (i) and (ii) hold. Let $\tilde{\Delta}_{i(s+1)}$ be the first interval after $\tilde{\Delta}_{i(s)}$ that contains points $j_m \neq \sigma_1, \dots, \sigma_s$ with $x_{m+1} = t_\nu$. Set

$$\sigma_{s+1} = j_{m_{s+1}} = \min\{j_m \in \tilde{\Delta}_{i(s+1)} \colon x_{m+1} = t_\nu,\ j_m \neq \sigma_1, \dots, \sigma_s\}. \tag{90}$$

Fix $j \in J'_k$ and set

$$\mu(i) = \max\{\mu \colon\ j_\mu \in \tilde{\Delta}_{i+1}\}, \quad i = 1, \dots, k_\nu - 1, \tag{91}$$

and

$$\varphi(i) = \sum_{m \in M_i} x_m, \quad M_i = \{m \colon 0 < x_m < t_\nu,\ 1 \leqslant m \leqslant \mu(i) + 1\}. \tag{92}$$

Next we consider a technical statement that plays a crucial role in the proof of Proposition 9.

Lemma 10. *If $\tilde{\Delta}_{i_p}$, $p = 1, 2, \ldots$, are the intervals for which no σ_s is chosen then we have*

$$\varphi(i_p) \geqslant p t_\nu / 2. \tag{93}$$

Proof. Obviously it is enough to prove that

$$\varphi(i_1) \geqslant t_\nu / 2 \tag{94}$$

and

$$\varphi(i_{p+1}) - \varphi(i_p) \geqslant t_\nu / 2, \quad p = 1, 2, \ldots. \tag{95}$$

Now we prove (94). If $i_1 = 1$ then there is no $j_m \in \tilde{\Delta}_1$ with $x_{m+1} = t_\nu$ (otherwise it would have been chosen as σ_1), so in particular $x_{\mu(1)+1} < t_\nu$. Put $m_1 = 0$ if there is no j_m with $j_m \leqslant a_1 - t_\nu/2$; otherwise set

$$m_1 = \max\{m\colon\ j_m \leqslant a_1 - t_\nu/2\}.$$

Then we obtain (since $x_{m_1+1} \leqslant t_\nu$)

$$\varphi(1) \geqslant \sum_{m_1+2}^{\mu(1)+1} x_m = j_{\mu(1)+1} - j_{m_1+1} \geqslant a_2 - (a_1 - t_\nu/2 + x_{m_1+1}) \geqslant t_\nu/2,$$

thus (94) holds.

Consider now the case where $i_1 = s > 1$. We know that $\sigma_1 \in \tilde{\Delta}_1, \ldots, \sigma_{s-1} \in \tilde{\Delta}_{s-1}$ have been chosen, but it is impossible to choose next σ_s in $\tilde{\Delta}_s$. Thus there is no $j_m \in \Delta_s$ with $x_{m+1} = t_\nu$, so in particular $0 < x_{\mu(s)+1} < t_\nu$.

Assume that on the contrary,

$$\varphi(i_1) < t_\nu/2. \tag{96}$$

We are going to prove by reverse induction in λ the following

Claim. *For every $\lambda = 1, \ldots, s-1$*

$$\sigma_\lambda = j_{\mu(\lambda)} \in (a_{\lambda+1} - t_\nu/2, a_{\lambda+1}]; \tag{97}$$

$$\textit{there is no } j_m \neq \sigma_\lambda \textit{ such that } j_m \in \Delta_\lambda \textit{ and } x_{m+1} = t_\nu. \tag{98}$$

Proof. Fix $\lambda = s-1$ and consider $j_{\mu(s-1)}$. There are three cases:

(1) $x_{\mu(s-1)+1} < t_\nu$; then $\varphi(s) \geqslant j_{\mu(s)+1} - j_{\mu(s-1)}$ (because only negative or “small” steps $0 < x_m < t_\nu$ are available for the walk from $j_{\mu(s-1)}$ to $j_{\mu(s)+1}$). Thus

$$\varphi(s) \geqslant x_{\mu(s-1)+1} + \cdots + x_{\mu(s)+1} > j_{\mu(s)+1} - a_s \geqslant t_\nu$$

which contradicts to (96).

(2) If $x_{\mu(s-1)+1} = t_\nu$ and $j_{\mu(s-1)} \leqslant a_s - t_\nu/2$ then $\varphi(s) \geqslant j_{\mu(s)+1} - j_{\mu(s-1)+1}$ (because only negative or "small" steps $0 < x_m < t_\nu$ are available for the walk from $j_{\mu(s-1)+1}$ to $j_{\mu(s)+1}$). Thus

$$\varphi(s) \geqslant x_{\mu(s-1)+2} + \cdots + x_{\mu(s)+1} > a_{s+1} - (a_s + t_\nu/2) = t_\nu/2,$$

which contradicts to (96).

(3) $x_{\mu(s-1)+1} = t_\nu$ and $j_{\mu(s-1)} > a_s - t_\nu/2$; then $\sigma_{s-1} = j_{\mu(s-1)}$ (because otherwise $j_{\mu(s-1)}$ would be chosen as $\sigma_s \in \tilde{\Delta}_s$), so (97) holds for $\lambda = s-1$. There is no $j_m \neq \sigma_{s-1}$ with $j_m = t_\nu$ such that $j_m \in \Delta_{s-1}$ (otherwise such j_m would be chosen as σ_{s-1} or σ_s), thus (98) also holds for $\lambda = s-1$.

Next suppose that the inductive claim holds for $\lambda = \tilde{\lambda}, \ldots, s-1$, where $1 < \tilde{\lambda} \leqslant s-1$. We are going to prove that (97) and (98) hold for $\tilde{\lambda} - 1$. Consider $j_{\mu(\tilde{\lambda}-1)}$. The following three cases may occur:

(1) $x_{\mu(\tilde{\lambda}-1)+1} < t_\nu$; then exactly $s - \tilde{\lambda}$ steps equal to t_ν are available for the walk from $j_{\mu(\tilde{\lambda}-1)}$ to $j_{\mu(s)+1}$. Thus

$$\varphi(s) \geqslant j_{\mu(s)+1} - j_{\mu(\tilde{\lambda}-1)} \geqslant a_{s+1} - a_{\tilde{\lambda}} - (s - \tilde{\lambda})t_\nu \geqslant t_\nu,$$

which contradicts (96).

(2) $x_{\mu(\tilde{\lambda}-1)+1} = t_\nu$ and $j_{\mu(\tilde{\lambda}-1)} \leqslant a_{\tilde{\lambda}} - t_\nu/2$; then exactly $s - \tilde{\lambda}$ steps equal to t_ν are available for the walk from $j_{\mu(\tilde{\lambda}-1)+1}$ to $j_{\mu(s)+1}$. Thus

$$\varphi(s) \geqslant j_{\mu(s)+1} - j_{\mu(\tilde{\lambda}-1)+1} \geqslant a_{s+1} - (a_{\tilde{\lambda}} + t_\nu/2) - (s - \tilde{\lambda})t_\nu = t_\nu/2,$$

which contradicts (96).

(3) $x_{\mu(\tilde{\lambda}-1)+1} = t_\nu$ and $j_{\mu(\tilde{\lambda}-1)} > a_{\tilde{\lambda}} - t_\nu/2$ then $\sigma_{\tilde{\lambda}-1} = j_{\mu(\tilde{\lambda}-1)}$ (because otherwise $j_{\mu(\tilde{\lambda}-1)}$ would be chosen as $\sigma_{\tilde{\lambda}} \in \tilde{\Delta}_{\tilde{\lambda}}$), so (97) holds for $\lambda = \tilde{\lambda} - 1$. There is no $j_m \neq \sigma_{\tilde{\lambda}-1}$ with $j_m = t_\nu$ such that $j_m \in \Delta_{\tilde{\lambda}-1}$ (otherwise such j_m would be chosen as $\sigma_{\tilde{\lambda}-1}$ or $\sigma_{\tilde{\lambda}}$), thus (98) also holds for $\lambda = \tilde{\lambda} - 1$. $\square$

Hence our induction claim holds for $\lambda = 1$. Moreover, there is no $j_m \neq \sigma_1$, $j_m \in \tilde{\Delta}_1$ with $x_{m+1} = t_\nu$ (otherwise it would be chosen as σ_1 or σ_2). Let $m_1 = \max\{m\colon\ j_m \leqslant a_1 - t_\nu/2\}$; then $j_{m_1+1} = j_{m_1} + x_{m_1+1} \leqslant a_1 + t_\nu/2$. On the other hand, for the walk from j_{m_1+1} to $j_{\mu(s)+1}$ there are exactly $s-1$ steps equal to t_ν, therefore

$$\varphi(s) \geqslant j_{\mu(s)+1} - j_{m_1+1} - (s-1)t_\nu \geqslant a_{s+1} - (a_1 + t_\nu/2) - (s-1)t_\nu = t_\nu/2.$$

This estimate contradicts (96), hence (94) is proved.

We omit the proof of (95) because it is essentially the same—one should only make the appropriate changes of the formulas like writing $\varphi(i_{p+1}) - \varphi(i_p)$ instead of $\varphi(s)$. Lemma 10 is proven. □

Now we prove that for every $j \in J_k'$ the construction can be carried on $\tilde{k}$ steps, so the map σ is well defined. Indeed, fix an arbitrary $j \in J_k'$. Let k_2 be the number of intervals $\tilde{\Delta}_i$ for which our construction does not correspond an σ_t. By Lemma 10 we have

$$\varphi(i_{k_2}) \geqslant k_2 \frac{t_\nu}{2},$$

so the number of "small positive steps" $x_m < t_\nu$ is greater or equal to $k_2 t_\nu / 2t_{\nu-1}$. On the other hand, by (84) the same number does not exceed

$$k - k' < (k - k_\nu)(1 + \varepsilon_\nu) = (k - k_\nu)\frac{t_\nu}{t_\nu - t_{\nu-1}}.$$

Therefore

$$k_2 \frac{t_\nu}{2t_{\nu-1}} < (k - k_\nu)\frac{t_\nu}{t_\nu - t_{\nu-1}},$$

so we obtain

$$k_2 \leqslant [(k - k_\nu) 2\varepsilon_\nu].$$

Hence there are at least

$$\tilde{k} = k_\nu - 1 - [(k - k_\nu) 2\varepsilon_\nu]$$

intervals $\tilde{\Delta}_i$ for which our construction will yield corresponding σ_i.

It remains to prove part (iii) of Proposition 9: if for every $j \in J_k'$ we set $\tau(j)$ to be the complementary $(k - \tilde{k})$-tuple to $\sigma(j)$ then $\tau(\cdot)$ is one-to-one correspondence. Indeed, suppose that $j, j' \in J_k'$ and $\tau(j) = \tau(j')$.

Now we prove by induction in s that

$$j_1 = j_1', \qquad \ldots, \qquad j_s = j_s'.$$

By the construction of σ we have $j_1 \neq \sigma_1(j)$ and $j_1' \neq \sigma_1(j')$, thus

$$j_1 = \tau_1(j) = \tau_1(j') = j_1'.$$

Assume that the induction claim holds for some s, $1 \leqslant s < k$. We have $j_{s+1} = j_s + x_{s+1}$ and $j_{s+1}' = j_s' + x_{s+1}'$. If $x_{s+1} < t_\nu$ and $x_{s+1}' < t_\nu$ then from the construction of σ it follows (for some t) that $j_{s+1} = \tau_t(j) = \tau_t(j') = j_{s+1}'$. In the case where $x_{s+1} = x_{s+1}' = t_\nu$ it is obvious that $j_{s+1} = j_{s+1}'$.

So, it remains to consider the case where only one of the steps x_{s+1}, x'_{s+1} equals t_ν, say $x_{s+1} = t_\nu$, but $x'_{s+1} < t_\nu$. Then either for some t we have $j_{s+1} = \tau_t(j)$ and $j'_{s+1} = \tau_t(j')$ (which is impossible because $\tau(j) = \tau(j')$), or for some t and t' we have

$$j'_{s+1} = \tau_{t'}(j'), \qquad j_{s+1} = \sigma_t(j)$$

(which is also impossible because then one can easily see by the construction of σ that $\tau_t(j) \neq \tau_{t'}(j')$). $\square$

Step 9. Now we estimate Z_k, $k_{\nu+4} \leqslant k \leqslant 5k_\nu$. By Step 8 we have

$$Z_k = \sum_{j \in J'_k} \frac{1}{|n^2 - j_1^2| \cdots |n^2 - j_k^2|} \leqslant Z_k^\sigma Z_k^\tau, \tag{99}$$

where

$$Z_k^\sigma = \min_{j \in J'_k} \frac{1}{|n^2 - \sigma_1^2| \cdots |n^2 - \sigma_{\tilde{k}}^2|}, \quad (\sigma_1, \ldots, \sigma_{\tilde{k}}) = \sigma(j),$$

and

$$Z_k^\tau = \sum_{j \in J'_k} \frac{1}{|n^2 - \tau_1^2| \cdots |n^2 - \tau_{k-\tilde{k}}^2|}, \quad (\tau_1, \ldots, \tau_{k-\tilde{k}}) = \tau(j).$$

Taking into account that the map $\tau : J' \to \mathbb{Z}^{k-\tilde{k}}$ is an injection we obtain by Lemma 4 that

$$Z_k^\tau \leqslant \left(\sum_{m \neq \pm n} \frac{1}{|n^2 - m^2|} \right)^{k-\tilde{k}} \leqslant \left(\frac{4 + 4\log 2n}{2n} \right)^{k-\tilde{k}}. \tag{100}$$

In order to estimate Z_k^σ let us recall that by the construction of $\sigma(j)$ we have

$$\sigma_i \in \tilde{\Delta}_i = (a_{m(i)} - t_\nu/2, a_{m(i)+1}], \quad i = 1, \ldots, \tilde{k}.$$

From every $\tilde{k}$-tuple $\sigma(j)$ one can construct a $k_\nu - 1$-tuple $d(j) = (d_1(j), \ldots, d_{k_\nu - 1})$ such that

$$d_m(j) \in \tilde{\Delta}_m, \quad m = 1, \ldots, k_\nu - 1,$$

by adding $k_\nu - 1 - \tilde{k}$ new points (chosen to belong to the intervals for which no $\sigma_i(j)$ was attached). Obviously we have

$$\left(n^2 - \sigma_1^2\right) \cdots \left(n^2 - \sigma_{\tilde{k}}^2\right) \geqslant \left(\frac{1}{n^2} \right)^{k_\nu - 1 - \tilde{k}} \left(n^2 - d_1^2\right) \cdots \left(n^2 - d_{k_\nu}^2\right). \tag{101}$$

Next we estimate $\prod_i (n^2 - d_i^2)$. Let

$$m^* = \max\{m\colon\ a_m \leqslant 0\}.$$

There are two cases:

(a) k_ν is an even number; then $m^* = k_\nu/2$ and $a_{m^*} = -t_\nu/2$;
(b) k_ν is an odd number; then $m^* = (k_\nu + 1)/2$ and $a_{m^*} = 0$.

The function $x \to n^2 - x^2$ is even, increasing on $[-n, 0]$ and decreasing on $[0, n]$. Therefore in the case (a) we have

$$n^2 - d_i^2 > n^2 - a_{i-1}^2 \quad \text{if } i \leqslant m^*, \qquad n^2 - d_i^2 > n^2 - a_{i+1}^2 \quad \text{if } i > m^*,$$

thus

$$\prod_{i=1}^{k_\nu - 1} \bigl(n^2 - d_i^2\bigr) > \bigl(n^2 - d_1^2\bigr) \prod_{m=1}^{m^*-1} \bigl(n^2 - a_m^2\bigr) \prod_{m=m^*+2}^{k_\nu - 1} \bigl(n^2 - a_m^2\bigr).$$

Taking into account that $\prod_{m=1}^{k_\nu-1}(n^2 - a_m^2) = (k_\nu!)^2 t_{k_\nu}^{2k_\nu}$ we obtain (since $n^2 - d_1^2 > n$, $n^2 - a_{m^*}^2 \leqslant n^2, n^2 - a_{m^*+1}^2 < n^2$)

$$\prod_{i=1}^{k_\nu - 1} \bigl(n^2 - d_i^2\bigr) > n^{-3} (k_\nu!)^2 t_{k_\nu}^{2k_\nu}. \tag{102}$$

In the case (b) we have

$$n^2 - d_i^2 > n^2 - a_{i-1}^2 \quad \text{if } i < m^*, \qquad n^2 - d_i^2 > n^2 - a_{i+1}^2 \quad \text{if } i \geqslant m^*,$$

and the same argument as in (a) shows that (102) holds.

By (101) and (102) we obtain

$$\bigl(n^2 - \sigma_1^2\bigr) \cdots \bigl(n^2 - \sigma_{\tilde{k}}^2\bigr) \geqslant n^3 \left(\frac{1}{n^2}\right)^{k_\nu - 1 - \tilde{k}} (k_\nu!)^2 t_{k_\nu}^{2k_\nu}$$

hence by (86)

$$Z_k^\sigma \leqslant \frac{n^{2[2(k-k_\nu)\varepsilon_\nu]+3}}{(k_\nu!)^2 t_{k_\nu}^{2k_\nu}}. \tag{103}$$

Now (100), (103), and (86) imply

$$Z_k \leqslant Z_k^\sigma Z_k^\tau \leqslant \frac{n^{2[2(k-k_\nu)\varepsilon_\nu]+3}}{(k_\nu!)^2 t_{k_\nu}^{2k_\nu}} \left(\frac{4 + 4\log 2n}{2n}\right)^{k - k_\nu + 1 + [2(k-k_\nu)\varepsilon_\nu]}. \tag{104}$$

Step 10. This is the final step in the proof of Theorem 7. Taking into account (67) we obtain from (80), (85), and (104)

$$S'_k \leqslant S_{k_\nu} n^{2[2(k-k_\nu)\varepsilon_\nu]+3} (2n)^{[(k-k_\nu)\varepsilon_\nu]} \left(\frac{4+4\log 2n}{2n}\right)^{k-k_\nu+1+[2(k-k_\nu)\varepsilon_\nu]}$$
$$\leqslant S_{k_\nu} (2n)^{3[2(k-k_\nu)\varepsilon_\nu]+3} \left(\frac{4+4\log 2n}{2n}\right)^{k-k_\nu+1}.$$

Choose ν_1 so that $4+4\log 2n \leqslant \frac{1}{2}(2n)^{1/5}$ for $n = n_\nu$ with $\nu \geqslant \nu_1$; then

$$S'_k \leqslant (2n)^{h(k)} 2^{-(k-k_\nu+1)} S_{k_\nu},$$

where

$$h(k) = 3[2(k-k_\nu)\varepsilon_\nu] + 3 - \frac{4}{5}(k-k_\nu+1).$$

Next we show that $h(k) \leqslant -1$ for large enough ν. Indeed, by (60) and (84) $\varepsilon_\nu \to 0$, therefore there is ν_2 such that

$$\varepsilon_\nu < 1/40 \quad \text{for } \nu > \nu_2.$$

Therefore we have for $\nu > \nu_2$

$$h(k_\nu + 4) = 3[8\varepsilon_\nu] - 1 = -1.$$

If $k \geqslant k_\nu + 5$ then

$$h(k) \leqslant 6(k-k_\nu)\varepsilon_\nu + 3 - \frac{4}{5}(k-k_\nu+1) \leqslant 5\left(\frac{6}{40} - \frac{4}{5}\right) + \frac{11}{5} < -1.$$

Therefore

$$S'_k \leqslant (2n)^{-1} 2^{-(k-k_\nu+1)} S_{k_\nu}, \quad k \geqslant k_\nu + 4,$$

hence we obtain

$$\sum_{k_\nu+4}^{5k_\nu} S'_k \leqslant \frac{1}{2n} S_{k_\nu}. \tag{105}$$

This estimate, together with the estimates (71) in Step 3, (75) in Step 5, (76) in Step 6 prove that $\beta_{k_\nu}(n, z_n)$ gives the main part of $\beta(n, z_n)$. In view of (70) this accomplishes the proof of Theorem 7. $\square$

5. Comments

(1) Consider weights of the form

$$\omega(j) = \exp(\varphi(j)), \quad j \in \mathbb{Z}, \tag{106}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an even convex function with $\varphi(0) = 0$. Then for every $k \geqslant 2$

$$\varphi\left(\frac{1}{k}x_1 + \cdots + \frac{1}{k}x_k\right) \leqslant \frac{1}{k}\varphi(x_1) + \cdots + \frac{1}{k}\varphi(x_k),$$

thus

$$k\varphi\left[\frac{1}{k}(x_1 + \cdots + x_k)\right] \leqslant \varphi(x_1) + \cdots + \varphi(x_k). \tag{107}$$

Set for any $\gamma > 0$

$$\Omega_\gamma(m) = \inf_{k \geqslant 1} |m|^{\gamma k - 1} \exp[k\varphi(m/k)]. \tag{108}$$

One may compute $\Omega_\gamma(m)$ explicitly for concrete functions φ; this was done in Lemma 6 in case where $\varphi(j) = a|j|^b$, $b > 1$.

The following statement generalizes Theorem 5.

Theorem 11. *Suppose $v(x) = \sum_n V(2n) \exp(2n\pi \mathrm{i} x)$ is a periodic potential, of period* 1, *such that*

$$\|V\|_\omega = \sup |V(2n)|\omega(2n) < \infty, \tag{109}$$

where ω is the weight (106). *If d_n, $n \in \mathbb{N}$, are the diameters of spectral triangles that corresponds to v then*

$$\sup_n d_n \Omega_\gamma(2n) < \infty, \quad \forall \gamma \in (0, 1). \tag{110}$$

Proof. By (108) we have $\Omega_\gamma(m) \leqslant \omega(m)$ since in the right-hand side of (108) we take inf of an expression that equals ω for $k = 1$, therefore

$$\sup |V(2n)|\Omega_\gamma(2n) \leqslant \sup |V(2n)|\omega(2n) = \|V\|_\omega.$$

So, in view of (21) and Proposition 2, it is enough to prove that

$$\sup_{|n| \geqslant n_0} |S(n)|\Omega_\gamma(2n) < \infty, \tag{111}$$

where $S(n) = \sum_{k=1}^{\infty} S_k(n)$ and $S_k(n)$ are defined by (34). By (107), with $x_1 = -n + j_1$, $x_2 = j_2 - j_1, \ldots, x_k = j_k - j_{k-1}, x_{k+1} = n - j_k$, we obtain

$$\exp\bigl((k+1)\varphi\bigl(2n/(k+1)\bigr)\bigr) \leqslant \omega(n + j_1)\omega(j_2 - j_1)\cdots\omega(j_k - j_{k-1})\omega(n - j_k),$$

therefore

$$\begin{aligned}\sigma_k(n) &:= |S_k(n)|\exp\bigl((k+1)\varphi\bigl(2n/(k+1)\bigr)\bigr)\\ &\leqslant \sum_{j_1,\ldots,j_k \neq \pm n} \frac{\sigma(n+j_1)\sigma(j_2-j_1)\cdots\sigma(j_k-j_{k-1})\sigma(n-j_k)}{|n^2 - j_1^2||n^2 - j_2^2|\cdots|n^2 - j_k^2|},\end{aligned}$$

where

$$\sigma(s) = |V(s)|\omega(s) \leqslant \|V\|_\omega, \quad s \in \mathbb{Z}.$$

From here it follows

$$\sigma_k(n) \leqslant \|V\|_\omega^{k+1} \sum_{j_1,\ldots,j_k \neq \pm n} \frac{1}{|n^2 - j_1^2||n^2 - j_2^2|\cdots|n^2 - j_k^2|},$$

thus by Lemma 4

$$\sigma_k(n) \leqslant \|V\|_\omega^{k+1} \left(\sum_{j \neq \pm n} \frac{1}{|n^2 - j^2|}\right)^k \leqslant \|V\|_\omega^{k+1} \left(4\frac{1 + \log(2n)}{2n}\right)^k. \tag{112}$$

Then by (108) and (112) we have

$$\begin{aligned}\sum_{k=1}^{\infty} |S_k(n)|\Omega_\gamma(2n) &\leqslant \sum_{k=1}^{\infty} |S_k(n)|\exp\bigl((k+1)\varphi\bigl(2n/(k+1)\bigr)\bigr)(2n)^{\mu(k+1)-1}\\ &\leqslant \sum_{k=1}^{\infty} \sigma_k(n)(2n)^{\mu k} \leqslant \sum_{k=1}^{\infty} \|V\|_\omega^{k+1}\left(4\frac{1+\log(2n)}{(2n)^{1-\mu}}\right)^k.\end{aligned}$$

Choose n_μ so that for $n \geqslant n_\mu$

$$4\|V\|_\omega \frac{1 + \log(2n)}{(2n)^{1-\mu}} \leqslant \frac{1}{2}. \tag{113}$$

Then by (113) we obtain

$$\sup_{n \geqslant n_\mu} \sum_{k=1}^{\infty} |S_k(n)|\Omega_\gamma(2n) \leqslant \|V\|_\omega \sum_{k=1}^{\infty} \frac{1}{2^k} = \|V\|_\omega < \infty. \qquad \Box$$

(2) In view of Theorems 5 and 7 the following natural questions arises:
Let $v(x) = \sum V(m)\exp(2\pi \mathrm{i} mx)$ be an entire function, i.e.,

$$|V(2m)| \leqslant C(a)\exp(-a|m|), \quad m \in \mathbb{Z},\ \forall a > 0. \tag{114}$$

If $v(x)$ is real-valued, suppose that the spectral gap sequence (γ_n) decays superexponentially in such a way that with $b > 1$

$$\gamma_n \leqslant C_1 \exp\bigl(-C_2 n(\log n)^{1-1/b}\bigr), \quad \exists C_1, C_2 > 0. \tag{115}$$

Question 1. Is it true that (115) together with (114) imply

$$\exists C_3, C_4\colon \quad |V(2m)| \leqslant C_3 \exp\bigl(-C_4|m|^b\bigr)? \tag{116}$$

If $v \in (114)$ is complex-valued we assume that (recall (30))

$$d_n \leqslant C_1 \exp\bigl(-C_2 n(\log n)^{1-1/b}\bigr), \quad \exists C_1, C_2 > 0, \tag{117}$$

and ask the same question:

Question 2. Is it true that (117) together with (114) imply (116)?

Similar questions have been answered positively in [2,3] (respectively for real-valued potentials and complex-valued potentials) in the case of subexponential Gevrey type decay rate of (γ_n) or (d_n). It is shown there that *if* (d_n) *has a Gevrey type decay, that is*

$$d_n \leqslant C\exp\bigl(-an^b\bigr), \quad a > 0,\ b \in (0,1),$$

then the L^2*-potential* v *belongs to the same Gevrey space, that is*

$$|V(2k)| \leqslant C_1 \exp\bigl(-a|k|^b\bigr), \quad k \in \mathbb{Z}.$$

The stronger assumption (114) instead of a simple requirement $v \in L^2$ is essential: finite-zone (real-valuead) potentials w satisfy conditions (115) and (117) because $\gamma_n = d_n = 0$ for $n \geqslant N$, but they are not entire functions. See further discussion on that phenomenon in [2], Section 5.3.

The proofs in [2,3] have two basic elements: “geometric” one, that leads to a special nonlinear relation (in some sequence spaces, with $|\eta_n| \asymp d_n$)

$$\eta_n = V(2n) + \beta(V; n, z_n) \tag{118}$$

(e.g., see Proposition 3 for the real-case statement), and analytic one, where this nonlinear relation is analyzed in order to prove the necessary a priori estimates of $(V(2m))$.

Certainly, in order to solve our Questions 1 and 2 one can always use the “geometric” results from [2,3] because they do not depend on the weight spaces one considers. But it turns out that the analysis of (118) is more difficult in the case of superexponential decay rate of (η_n) and $V(2m)$ due to the fact that in this case, as the proofs of Theorems 5 and 7 show, $V(2n)$ is not the main term of the right-hand side of (118).

(3) If the potential $v(x)$ is $b\cos 2\pi x,\ b>0$, the sharp estimates of spectral gaps have been given by E. Harrell [6] and J. Avron and B. Simon [1]; it has been shown that in this case

$$\gamma_n = 8\pi^2\left(\frac{b}{4\pi^2}\right)^n \frac{1}{((n-1)!)^2}\bigl(1+O(1/n^2)\bigr).$$

A. Grigis [5] considered the case of real valued trigonometric polynomials of the form

$$v(x) = \cos 2\pi Nx + \sum_{m=-N+1}^{N-1} V(2m)\exp(2\pi \mathrm{i} mx). \tag{119}$$

He derived a polynomial $Q(t)$ of degree $N-1$ whose coefficients depend algebraically on those of $v(x)$; the construction of this polynomial led to a construction of an asymptotic formula for γ_n. Therefore, almost any polynomial (119) has nonzero coefficients in asymptotic terms; however, for specific (119) it is not easy to say which of these coefficients is different from zero.

By using the results and constructions of [2,3,8,9] we can give the upper-estimates of the diameters of spectral triangles in the case of *any* trigonometric polynomial potentials (119). The following statement is true:

Proposition 12. *Suppose* $v(x) = \sum_{m=-T}^{T} V(2m)\exp(2\pi \mathrm{i} mx)$, *where* $|V(2m)| \leqslant Q$. *Let* d_n *be the diameters of the corresponding spectral triangles*; *then*

$$d_n \leqslant \frac{(C_1TQ)^{2n/T}}{n^{4n/T}}, \quad C_1 > 1, \tag{120}$$

where C_1 *is an absolute constant.*

Remark. Of course, A. Grigis results [5] show that (120) is sharp in the class of polynomials of degree T.

Proof. Let $2n > T$; then $V(2n) = 0$. Therefore, in view of Proposition 1, formula (21) and Proposition 2 we have for $n \geqslant n_0$

$$d_n \leqslant K\sum_{k=1}^{\infty} S_k(n),$$

where K is an absolute constant and

$$S_k(n) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{|V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)|}{|(n^2-j_1^2)\cdots(n^2-j_k^2)|}. \tag{121}$$

Let J_k be the set of all k-tuples $j = (j_1, \dots, j_k)$, $j_i \neq \pm n$, such that the corresponding terms of (121) do not vanish. Observe that if

$$V(n + j_1)V(j_2 - j_1)\cdots V(j_k - j_{k-1})V(n - j_k) \neq 0 \tag{122}$$

then

$$\begin{aligned} &|n + j_1| \leqslant T, \qquad |j_2 - j_1| \leqslant T, \qquad \dots, \\ &|j_k - j_{k-1}| \leqslant T, \qquad |n - j_k| \leqslant T, \end{aligned} \tag{123}$$

so

$$2n \leqslant \bigl|(n + j_1) + (j_2 - j_1) + \cdots + (j_k - j_{k-1}) + (n - j_k)\bigr| \leqslant (k + 1)T.$$

Thus by (121) $S_k(n) = 0$ for $k + 1 < 2n/T$ and we have

$$d_n \leqslant C \sum_{k=k_0}^{\infty} S_k(n), \quad k_0 = [2n/T] - 1. \tag{124}$$

From (123) it follows that there are at most $(2T)^k$ nonzero terms in $S_k(n)$. Indeed, since $|n - j_1| \leqslant T$ we have at most $2T$ choices for $j_1 \neq 0$; then, if j_1 is fixed, since $|j_2 - j_1| \leqslant T$ we have at most $2T$ choices for j_2, and so on. Therefore we obtain for $k \geqslant k_0$

$$S_k(n) \leqslant Q^{k+1}(2T)^k \Pi_k^{-1}, \tag{125}$$

where

$$\Pi_k = \min\left\{ \prod_{\nu=1}^{k} \bigl(n^2 - j_\nu^2\bigr):\ (j_1, \dots, j_k) \in J_k \right\}. \tag{126}$$

Let $k \geqslant k_0$. Fix a $j \in J_k$ and regard j as a walk from $-n$ to n. By (123) the length of every step of j does not exceed T, therefore every subinterval of $(-n, n)$ of the form $(a, a+T]$ contains at least one of the points $j_1, \dots, j_k$. Set $a_m = -n + mT$, $m = 1, \dots, k_0$, and choose points

$$j_{i(m)} \in (a_m, a_{m+1}], \quad m = 1, \dots, k_0 - 1.$$

Moreover, if $m^* = \max\{m\colon\ a_m \leqslant 0\}$ then we have: $n^2 - j_{i(m)}^2 \geqslant n^2 - a_m^2$ for $m < m^*$; $n^2 - j_{i(m)}^2 \geqslant n^2 - a_{m+1}^2$ for $m > m^*$; either $n^2 - j_{i(m)}^2 \geqslant n^2 - a_{m^*}^2$, or $n^2 - j_{i(m)}^2 \geqslant n^2 - a_{m^*+1}^2$ for $m = m^*$. From these inequalities it follows

$$\prod_{m=1}^{k_0-1} \bigl(n^2 - j_i^2\bigr) \geqslant \frac{1}{n^2} \prod_{m=1}^{k_0} \bigl(n^2 - a_m^2\bigr) = \frac{1}{n^2}(k_0!)^2 T^{2k_0},$$

therefore

$$\prod_{m=1}^{k}\left(n^2 - j_i^2\right) \geqslant n^{k-k_0-1}(k_0!)^2 T^{2k_0}. \tag{127}$$

Now (125)–(127) lead to

$$S_k(n) \leqslant \frac{Q^{k+1}(2T)^k}{n^{k-k_0-1}(k_0!)^2 T^{2k_0}} = \left(\frac{2TQ}{n}\right)^{k-k_0} \frac{n 2^{k_0} Q^{k_0+1}}{(k_0!)^2 T^{k_0}}. \tag{128}$$

Let $n \geqslant 4TQ$; then by (128)

$$\sum_{k=k_0}^{\infty} S_k(n) \leqslant \frac{n(2Q)^{k_0+1}}{(k_0!)^2 T^{k_0}}. \tag{129}$$

Since $k_0 + 1 = [2n/T]$ the Stirling formula shows that (124) and (129) imply (120) with some constant $C_1 > 1$. □

(4) Consider the Dirac operator

$$L = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix},$$

where φ, ψ are periodic, of period 1, L^2-functions. If

$$F = \begin{pmatrix} f \\ g \end{pmatrix} \in H^1 \times H^1 \subset L^2[0,1] \times L^2[0,1],$$

then LF is well-defined. We consider boundary conditions of three types:

$$\begin{aligned} &\mathrm{Per}^+\colon\quad F(0) = F(1), \qquad \mathrm{Per}^-\colon\quad F(0) = -F(1), \\ &\mathrm{Dir}\colon\quad f(0) = g(0), \qquad f(1) = g(1). \end{aligned}$$

Then for every $n \in \mathbb{Z}$ with large enough $|n|$ there are three eigenvalues $\{\lambda_n^+, \lambda_n^-, \mu_n\}$ close to πn, namely μ_n is an eigenvalue for Dir, and λ_n^+, λ_n^- are eigenvalues for Per^+ if n is even and for Per^- if n is odd. As in the case of Schrödinger operator (30) we define

$$d_n = \max\left\{\left|\lambda_n^+ - \lambda_n^-\right|, \left|\lambda_n^+ - \mu_n\right|, \left|\lambda_n^- - \mu_n\right|\right\}$$

as *a diameter of this spectral triangle.*

The relationship between smoothness of the potentials and decay rates of spectral gaps (or diameters of spectral triangles) has been studied in [12,14,15] for Dirac operators. The general scheme suggested in [8] can be used (as B. Grébert, T. Kappeler and B. Mityagin observed, and elaborated in [4] to get estimates of $|\gamma_n|$ in the case of analytic potentials) in analysis of Dirac operators as well. The further results in [9] and ours, in [2,3] and in this paper, could be reconstructed and adjusted to Dirac operators. In particular analogues of Theorem 5 and Proposition 12 hold. We'll present these results elsewhere.

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